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Maximal Structures of Determinate Propositions in Quantum Mechanics

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I formulate and answer some questions concerning maximal structures of 'determinate' quantum propositions, i.e., maximal structures of propositions that can be taken as having definite (but perhaps unknown) truth values for a given quantum state. The basic constraint on such structures is the Kochen and Specker 'no-go' hidden-variables theorem, which demonstrates that no value assignment to certain finite sets of observables can preserve the functional relations between commuting observables. The problem I want to consider is how large we can take the set of determinate observables without violating the functional relationship constraint. I show how to construct maximal determinate sublattices of quantum propositions that are unique, subject to certain constraints, and I comment on the relevance of this 'go' theorem for the interpretation of quantum mechanics.

The dynamical variables of a classical mechanical system form a commutative algebra, and the subalgebra of idempotent dynamical variables—the two-valued variables representing the possible outcomes of yes–no experiments on the system, or the truth values of propositions of the system—form a Boolean algebra. Classical states select ultrafilters in this Boolean algebra, i.e., maximal sets of propositions that are simultaneously true of the system, or maximal sets of properties that are simultaneously determinate for the system (generated by the assignment of a determinate value to every dynamical variable).

The dynamical variables of a quantum mechanical system form a noncommutative algebra, and the subalgebra of idempotents, represented by the projection operators on the Hilbert space of the system, form a non-Boolean algebra. The quantum state is understood as defining probabilities on this non-Boolean algebra of yes–no experiments, or propositions, or properties

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of the system. Since there are no quantum states that are dispersion free for all dynamical variables, no quantum state assigns a determinate value simultaneously to every dynamical variable. The dynamical variables are referred to as 'observables,' and the probability defined by the quantum state for a range of values of an observable is interpreted, minimally, as the probability of finding the value of the observable in the range on measurement (or, equivalently, the probability of finding the value 'yes' for the corresponding yes–no measurement, or the probability that the corresponding proposition will be found to be true of the system on measurement).

The problem of the completeness of quantum mechanics, or the problem of 'hidden variables,' has always concerned the question of whether it is possible to define states that assign determinate values simultaneously to *all* the observables of a quantum mechanical system, in such a way that the probabilities defined by quantum states can be recovered as measures over the different possible sets of value assignments. Various 'no-go' theorems show that such states are impossible if the value assignments are required to satisfy certain constraints.

The constraint imposed by the Kochen and Specker theorem (Kochen and Specker, 1967) requires that the values assigned to a set of mutually commuting observables should preserve the functional relations satisfied by these observables. With sums and products defined for mutually commuting observables only, the observables of a quantum mechanical system form a partial algebra, and the idempotent observables or projection operators form a partial Boolean algebra. Kochen and Specker show that a necessary condition for the simultaneous assignment of values to all the observables of a quantum mechanical system satisfying the functional relationship constraint is that, for every pair of distinct projection operators a, b in the partial Boolean algebra, there exists a homomorphism h onto the 2-element Boolean algebra such that $h(a) \neq h(b)$.

The Kochen and Specker proof proceeds by showing that there are no 2-valued homomorphisms on the partial Boolean algebra of projection operators on a Hilbert space of three or more dimensions. The homomorphism condition requires that for every orthogonal triple of 1-dimensional projection operators or corresponding rays in H_3, one projection operator or ray is mapped onto 1 ('true') and the remaining two projection operators or rays are mapped onto 0 ('false'). This is shown to be impossible for the finite set of orthogonal triples of rays that can be constructed from 117 appropriately chosen rays in H_3: any assignment of 1's and 0's to this set of orthogonal triples satisfying the homomorphism condition involves a contradiction.

Hidden-variables theories that satisfy the Kochen and Specker constraint are termed 'noncontextual.' A nonmaximal (degenerate) observable A might commute with B and also with C, while B and C fail to commute. For example, A and B could form a complete commuting set (in which case both A and B are functions of a maximal or nondegenerate observable X), and A and C could form an incompatible complete commuting set (in which case A and C are functions of a maximal observable Y not commuting with X). The functional relationship constraint forces the hidden variables to assign the same value to A, whether A is considered as part of the complete commuting set or measurement context (A, B), and hence a function of X, or as part of the complete commuting set or measurement context (A, C), and hence a function of Y.

Bell (1966) argued that the general requirement of noncontextuality cannot be justified on physical grounds, but pointed out that a special case of noncontextuality is physically plausible. If A refers to an observable of a subsystem S of a system $S + S'$, and B and C refer to possibly noncommuting observables of a subsystem S' spacelike separated from S, then the requirement of noncontextuality, that the value of A in the context of B should be the same as the value of A in the context of C, becomes a locality condition. Bell's (1964) 'no-go' theorem shows that a general version of this locality condition cannot be satisfied.

Several authors have considered the problem of constructing the smallest set of observables that cannot be assigned values in such a way as to satisfy the noncontextuality or locality constraint. Kochen and Conway [see Mermin (1993)] have reduced the number of directions in H_3 required to generate a contradiction from value assignments satisfying the Kochen and Specker homomorphism condition from 117 to 31, and Peres (1991) has found a more symmetrical proof with 33 rays. Peres (1991) has reduced the number of rays to 24 in H_4, and Kernaghan (1994) has shown how to reduce this to 20 rays. Mermin (1993) proves a version of the Kochen and Specker theorem for nine observables in a Hilbert space of four or more dimensions, and a version of Bell's theorem for ten observables in a Hilbert space of eight or more dimensions.

The problem of how small we can make the set of observables and still generate a Kochen–Specker contradiction is interesting mathematically and important in revealing structural features of Hilbert space, but of no immediate significance for the interpretation of quantum mechanical probabilities. The problem I wish to consider here is in a sense the converse of this problem: How *large* can we take the set of observables *without* generating a Kochen–Specker contradiction, i.e., what are the maximal sets of observables that can be taken as having determinate (but perhaps unknown) values for a given quantum state? (By the remarks in the previous section, any hidden-variables theory satisfying the Kochen and Specker constraint will necessarily satisfy the locality condition.)

More precisely, if we consider the propositions of a quantum mechanical system as a lattice L (isomorphic to the lattice of projection operators or corresponding Hilbert space subspaces of the system), we know that we cannot assign truth values to all the propositions in L in such a way as to satisfy the Kochen and Specker constraint. That is, we cannot take all the propositions in L as determinately true or false if truth values are assigned subject to this constraint. So the probabilities defined by the quantum state cannot be interpreted epistemically and represented as measures over the different possible truth value assignments to *all* the propositions in L. We also know that any single observable can be taken as determinate for any quantum state (since the propositions associated with an observable generate a Boolean algebra), so we may suppose that fixing a quantum state represented by a ray $e(\psi)$ in H and an arbitrary observable R places restrictions on what propositions can be taken as determinate for e in addition to R-propositions.

Identifying all such maximal determinate sets of propositions or associated observables amounts, in effect, to a 'go' theorem for hidden variables, specifically a hidden-variables theory in which the value of a privileged determinate observable R plays the role of a hidden variable that, together with the quantum state, assigns truth values to the propositions in a maximal determinate sublattice $D(e, R)$ of L selected by R and the quantum state (or, equivalently, assigns values to a maximal determinate subset of observables of the system selected by R and the quantum state, not to all observables). The probabilities specified by the quantum state would then be interpreted as measures over the different possible truth value assignments to the propositions in $D(e, R)$ (a sublattice that changes as the quantum state evolves dynamically), not to all propositions in L.

One might suppose that the maximal determinate sublattices $D(e, R)$ are simply maximal Boolean sublattices of L associated with the different maximal observables R. But this is not the case. It can be shown that, for any quantum state represented by a ray e in an n-dimensional Hilbert space H and any observable R, the *unique* choice for $D(e, R)$, subject to certain constraints (see below), is the sublattice

$$L_{e_{r_1}e_{r_2}\cdots e_{r_k}} = \{e_{r_i},\ i = 1, \ldots, k\}'$$

the commutant in L of $\{e_{r_i},\ i = 1, \ldots, k\}$, where the k rays $e_{r_i} = (e \vee r_i^{\perp}) \wedge r_i$ are the nonzero projections of the quantum state e onto the $m \leq n$ eigenspaces r_i of R $(k \leq m)$.

I use the same symbol r_i here for eigenvalues of R and the associated eigenspaces. The symbol $\{\ \ \}'$ indicates the commutant in L of $\{\ \ \}$, the set of all operators that commute with the projectors in $\{\ \ \}$. The symbol $^{\perp}$ represents the orthocomplement. My previous proposal for a type of 'modal' interpretation (Bub, 1992a, b) was, in effect, to take $D(e, R)$ as $\cap\{e_{r_i}, r - e_{r_i}\}'$, where the intersection is taken over the k nonzero projections of e onto the m eigenspaces of R. [See Cassinelli and Lahti (1994) for a formal characterization of different versions of the modal interpretation.] Evidently, $\cap\{e_{r_i}, r - e_{r_i}\}' \subseteq \{e_{r_i}, e = 1, \ldots, k\}'$, because $\cap\{e_{r_i}, \mathrm{r} - \mathrm{e}_{r_i}\}' \subseteq \cap\{e_{r_i}, H - \mathrm{e}_{r_i}\}'$ and $\cap\{e_{r_i}, H - \mathrm{e}_{r_i}\}' = \{e_{r_i}, e = 1, \ldots, k\}'$.

The sublattice $L_{e_{r_1}e_{r_2}\cdots e_{r_k}}$ of L is generated by the atoms e_{r_i}, $i = 1, \ldots, k$, and the atoms represented by all the rays in the subspace $(e_{r_1} \vee e_{r_2} \vee \ldots e_{r_k})^{\perp}$ orthogonal to the subspace spanned by the e_{r_i}. Since the e_{r_i} are orthogonal, they are compatible and generate a Boolean sublattice of L. So

$$(e_{r_1} \vee e_{r_2} \vee \ldots e_{r_k})^{\perp} = (e_{r_1})^{\perp} \wedge (e_{r_2})^{\perp} \wedge \ldots (e_{r_k})^{\perp}$$

It follows that

$$L_{e_{r_1}e_{r_2}\cdots e_{r_k}} = L_{e_{r_1}} \cap L_{e_{r_2}} \cap \ldots L_{e_{r_k}}$$

because each $L_{e_{r_i}}$, $i = 1, \ldots, k$, is generated by the ray e_{r_i} and all the rays in the subspaces $(e_{r_i})^{\perp}$ orthogonal to e_{r_i}. The set of maximal observables associated with $L_{e_{r_1}e_{r_2}\cdots e_{r_k}}$ includes any maximal observable with k eigenvectors in the directions e_{r_i}, $i = 1, \ldots, k$. The full set of observables associated with $L_{e_{r_1}e_{r_2}\cdots e_{r_k}}$ includes any observable whose eigenspaces are spanned by rays in $L_{e_{r_1}e_{r_2}\cdots e_{r_k}}$.

I originally imposed two basic constraints on the sublattices $D(e, R)$ (see Bub, 1994):

(1)° *Truth condition.* Truth values can be assigned to all the propositions of $D(e, R)$, where each assignment of truth values is defined by a 2-valued map on $D(e, R)$ that reduces to a 2-valued homomorphism on each Boolean sublattice of $D(e, R)$.

(2)° *Probability condition.* The probabilities defined by e on the propositions of $D(e, R)$ can be represented as measures over the different possible truth value assignments to $D(e, R)$, i.e., as measures on a Kolmogorov probability space (X, F, μ), where X is the set of truth-value maps on $D(e, R)$, as defined in (i), and $\mu(\{h\colon h(a) = 1\}) = \mathrm{tr}(ea)$ for any proposition $a \in D(e, R)$. Here $\mathrm{tr}(ea)$, the trace of ea, is the probability assigned by the quantum state e to the proposition represented by the subspace a, and $\mu(\{h\colon h(a) = 1\})$ is the measure of the set of truth-value maps assigning the value 1 (i.e., true) to a.

I characterized the maximal determinate sublattices $D(e, R)$ as the maximal sublattices of L satisfying the conditions (1)° and (2)°, together with the following four conditions:

(3)° *Eigenstate condition.* If e is an eigenstate of R, then $D(e, R)$ contains the proposition e (the proposition represented by the 1-dimensional subspace e).

(4)° *Impossibility condition.* If $a \in D(e, R)$ and $a \leq e^{\perp}$, then $b \in D(e, R)$ if $b \leq a$.

(5)° *Refinement condition.* If R is a refinement of $R^{\#}$ (so that the eigenspaces of $R^{\#}$ are either the same as the eigenspaces of R or spans—suprema—of some of the eigenspaces of R), and the possible atoms[2] of $D(e^{\#}, R^{\#})$, for any states $e^{\#}$, are a proper subset of the possible atoms of $D(e, R)$, then $D(e, R) \subset D(e^{\#}, R^{\#})$, for all such states $e^{\#}$.

(6)° *Measurement condition.* If the unit vector in the ray representing the quantum state takes the form of a polar decomposition $\Sigma\, c_i\alpha_i \otimes \rho_i$ with respect to eigenvectors a_i of some observable A and eigenvectors r_i of the observable R, so that A-propositions become correlated with R-propositions in the quantum state, then $D(e, R)$ includes the Boolean algebra of A-propositions (i.e., all propositions represented by the eigenspaces of A).

Note that modal interpretations that exploit the polar decomposition theorem (e.g., Kochen, 1985; Dieks, 1994; Healey, 1989) appeal to the existence of a unique decomposition of the form $\Sigma\, d_i\beta_i \otimes \tau_i$, in terms of the eigenvectors of some observables B and T, when the coefficients $|d_i|$ are all distinct. The measurement condition here requires only that observables correlated with R, *when the state takes the polar form for* R, are determinate.

It now follows from the eigenstate condition and the impossibility condition that if $R = I$, the unit observable, then $D(e, I) \supseteq L_e$, where L_e is the sublattice generated by the atom e and the atoms represented by all the rays in the subspace orthogonal to e. It can be shown that L_e is maximal if the Hilbert space H is more than 2-dimensional. [For a proof, see Bub (1994).] So, with the exception of the 2-dimensional case, $D(e, I) = L_e$. Note that H must be at least 4-dimensional to allow representations of the quantum state that can be interpreted as measurements.

The refinement condition is motivated by the requirement that if R is a refinement of $R^{\#}$, then the transition from $D(e, R)$ to $D(e^{\#}, R^{\#})$ should simply involve the transformation of some of the possible atomic propositions in $D(e, R)$ to impossible atomic propositions in $D(e^{\#}, R^{\#})$, with the appropriate modification to the subspace of impossible propositions to conform to the impossibility condition. The latter condition requires that every ray in the subspace of impossible propositions in $D(e^{\#}, R^{\#})$ belongs to $D(e^{\#}, R^{\#})$, which

[2]The impossible propositions are the propositions in $D(e, R)$ assigned zero probability by the measure μ corresponding to e, and the possible propositions are the propositions in $D(e, R)$ assigned nonzero probabilities by μ.

means that $D(e^{\#}, R^{\#})$ contains some impossible propositions that do not belong to $D(e, R)$, and so $D(e, R) \subset D(e^{\#}, R^{\#})$.

Since every observable can be regarded as a refinement of the unit observable I, the refinement condition entails the special refinement condition: $D(e, R) \subset D(e^{\#}, I)$ if the possible atoms of $D(e^{\#}, I)$, for any states $e^{\#}$, are a proper subset of the possible atoms of $D(e, R)$. Since $D(e^{\#}, I) = L_{e^{\#}}$ for any $e^{\#}$, and contains only one possible atom represented by the ray $e^{\#}$, the special refinement condition requires that $D(e, R) \subset L_{e^{\#}}$ for any possible atom represented by a ray $e^{\#}$ in $D(e, R)$.

In the case that the Hilbert space H is a tensor product space with the observable R associated with a factor space, and the vector representative of the quantum state e takes a polar form $\sum c_i\alpha_i \otimes \rho_i$, where the ρ_i are eigenvectors of R, the measurement condition together with the impossibility condition entails that $D(e, R)$ contains elements corresponding to all the eigenspaces of any observable with k eigenvectors in the directions e_{r_i}, where the e_{r_i} are the nonzero projections of e onto the m eigenspaces of R. This follows because the measurement condition requires that $D(e, R)$ contains the propositions associated with the eigenspaces of any observable A with k eigenvectors α_i as well as R, and hence the propositions associated with the eigenspaces of an observable with k eigenvectors $\alpha_i \otimes \rho_i$ (the nonzero projections of $\sum c_i\alpha_i \otimes \rho_i$ onto the eigenspaces of R) and $n - k$ eigenvectors $\alpha_j \otimes \rho_k$, $j \neq k$, corresponding to tensor products of A-eigenvectors and R-eigenvectors assigned zero probability by e (where n is the dimensionality of H). The impossibility condition requires in addition that all atomic propositions represented by rays in the subspace spanned by the $n - k$ vectors assigned zero probability by e are in $D(e, R)$, which means that the propositions associated with all observables with eigenvectors coinciding on the k tensor products $\alpha_i \otimes \rho_i$ of A-eigenvectors and R-eigenvectors assigned nonzero probability by the quantum state belong to $D(e, R)$.

On the orthodox interpretation of quantum mechanics, an observable has a determinate value if and only if the state e is an eigenstate of the observable. So the determinate propositions in the state e are the propositions assigned probability 1 or 0 by e, i.e., the sublattice of propositions p in the set

$$\{p: e \leq p \text{ or } e \leq p^{\perp}\}$$

This is the sublattice $D(e, I) = L_e$, i.e., the orthodox interpretation in effect takes the privileged observable as the unit observable I.

The problem with the orthodox interpretation is that it leads to the measurement problem. Consider a model quantum mechanical universe consisting of a system S and a system M, which plays the role of a measuring instrument for observables of S. Let R represent the indicator or 'pointer' observable of M. A measurement interaction between S and M, say a unitary transformation that correlates eigenstates α_i of an observable A of S with eigenstates ρ_i of R, results in a state represented by a unit vector of the form $\psi = \Sigma_i c_i \alpha_i \otimes \rho_i$ (assuming initial pure states for S and M). Neither R-propositions nor A-propositions belong to the sublattice L_e when $e = e(\psi)$. In order to avoid the problem, we have to assume the projection postulate, that unitary evolution is suspended in the case of a measurement interaction, and the state of $S + M$ is projected onto the ray spanned by one of the unit vectors $\alpha_i \otimes \rho_i$ with probability $|c_i|^2$.

$L_{e_{r_1} e_{r_2} \cdots e_{r_k}}$ can be characterized as the sublattice of propositions p in the set[3]

$$\{p\colon e_{r_i} \leq p \text{ or } e_{r_i} \leq p^{\perp}, i = 1, \ldots, k\}$$

The projections of the ray $e(\psi)$ spanned by ψ onto the eigenspaces r_i of R are the rays $e_{r_i}(\psi)$ spanned by the unit vectors $\alpha_i \otimes \rho_i$. So for this state, the determinate sublattice contains propositions represented by the projection operators $a_i \otimes I_M$, where a_i here represents the projection operator onto the subspace spanned by the unit vector α_i, i.e., propositions corresponding to the eigenvalues of A. It follows that the propositions corresponding to the observable correlated with the pointer observable in the 'entangled' state arising from a unitary transformation representing a quantum mechanical measurement interaction are determinately true or false.

There are k Boolean homomorphisms on $L_{e_{r_1} e_{r_2} \cdots e_{r_k}}$ if the subspace $(e_{r_1} \vee e_{r_2} \vee \ldots e_{r_k})^{\perp}$ is more than 2-dimensional, where the ith homomorphism maps the proposition e_{r_i} onto 1. If the subspace $(e_{r_1} \vee e_{r_2} \vee \ldots e_{r_k})^{\perp}$ is less than 3-dimensional, there will also be Boolean homomorphisms that map this subspace onto 1 and each of the rays e_{r_i}, $i = 1, \ldots, k$, onto 0, but these Boolean homomorphisms will all be assigned zero measure by the measure m corresponding to e on the Kolmogorov probability space of Boolean homomorphisms on $L_{e_{r_1} e_{r_2} \cdots e_{r_k}}$ [since e is orthogonal to $(e_{r_1} \vee e_{r_2} \vee \ldots e_{r_k})^{\perp}$]. To generate the probabilities defined by e for the propositions in $L_{e_{r_1} e_{r_2} \cdots e_{r_k}}$ on the Kolmogorov probability space, the Boolean homomorphism (more precisely, the corresponding singleton subset) that maps e_{r_i} onto 1 is assigned measure $\mathrm{tr}(er_i)$, for $i = 1, \ldots, k$. So, if we take the sublattice $L_{e_{r_1} e_{r_2} \cdots e_{r_k}}$ as the determinate sublattice for the system $S + M$ in the state e, the probabilities defined by a quantum state for the eigenvalues of an observable A can indeed be interpreted as 'the probabilities of finding the different possible eigenvalues of A in a measurement of A.'

[3]This was pointed out by Rob Clifton (private communication).

At the August 1994 meeting of the International Quantum Structures Association in Prague, Rob Clifton proposed reformulating the 'truth condition' on the sublattices $D(e, R)$ in terms of 2-valued lattice homomorphisms, rather than the weaker requirement of 2-valued maps that reduce to 2-valued homomorphisms on each Boolean sublattice $D(e, R)$, and deriving the sublattices $L_{e_{r_1}e_{r_2}\cdots e_{r_k}}$ without any measurement condition. Clifton and I have now proved a uniqueness theorem for the sublattices $L_{e_{r_1}e_{r_2}\cdots e_{r_k}}$ on the basis of the following constraints (Bub and Clifton, n.d.):

(1) *Truth and probability.* $D(e, R)$ is an ortholattice admitting sufficiently many 2-valued homomorphisms h: $D(e, R) \rightarrow \{0, 1\}$ to recover the joint probabilities assigned by the state e to mutually compatible sets of elements $\{a_i\}_{i \in I}$, $a_i \in D(e, R)$, as measures on a Kolmogorov probability space (X, F, μ), where X is the set of 2-valued homomorphisms on $D(e, R)$ and $\mu(\{h: h(a_i) = 1, \text{ for all } i \in I\}) = \text{tr}(\Pi a)$.

(2) *e, R-Definability.* For any e and R, $D(e, R)$ is invariant under lattice homomorphisms (i.e., unitary and antiunitary transformations) that preserve e and R.

It is understood, of course, that R is determinate, i.e., that the eigenspaces r_i of R belong to $D(e, R)$. To avoid a problem with 2-dimensional subspaces, it is also assumed that if two systems are not 'entangled' by any interaction, then each system is characterized by its own determinate sublattice, where the determinate sublattice of a system is the restriction of the determinate sublattice of the composite system to the component system. With these constraints, it can be proved that the maximal determinate sublattices are just the sublattices $L_{e_{r_1}e_{r_2}\cdots e_{r_k}}$.

From the Bub–Clifton result, it follows that, *without introducing any measurement constraints on the determinate sublattices,* we can *derive* that the propositions corresponding to the observable correlated with the pointer observable in the 'entangled' state arising from a unitary transformation representing a quantum mechanical measurement interaction are determinately true or false. So we can *derive* the interpretation of the probabilities defined by a quantum state for the eigenvalues of an observable A as 'the probabilities of finding the different possible eigenvalues of A in a measurement of A.'

The uniqueness theorem characterizes a class of admissible interpretations of quantum mechanics. For example, Bohm's hidden-variables theory (Bohm, 1952) can be understood as a proposal for implementing an interpretation in which the privileged observable R is fixed as position in configuration space. With a fixed preferred observable R, which is now stipulated as always having a determinate value, the question arises as to the dynamics of 'value states' on the determinate sublattice $D(e, R)$ as the state e evolves in time, i.e., states defined by 2-valued homomorphisms on $D(e, R)$. Since the evolution of such states is completely determined by the evolution of e and of R, we want an equation of motion for the determinate values of R that will preserve the distribution of R-values specified by e, as e evolves in time according to Schrödinger's time-dependent equation of motion. It turns out that one possible choice for this dynamics reduces to Bohm's dynamics when R is position in configuration space (Bub, 1995).

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